Alg. geometry 1 (WiSe 2025-26) Sheet 8, Exercise 1

On some open $\mathcal{U} \subset \mathbb{P}^n$: $\varphi_{|\mathcal{U}} = [F_0 : \dots : F_m]_{|\mathcal{U}}$ $F_i \in K[x_0, \dots, x_n]$ hom. pol. with the same degree and without common zero on $\mathcal{U}$.

By shrinking $\mathcal{U}$ we can assume that $F_0, F_1, \dots, F_m$ do not have a common factor.

Claim 1: $\varphi = [F_0 : F_1 : \dots : F_m]$ on the whole $\mathbb{P}^n$.

Let $p \in \mathbb{P}^n$ (possibly outside $\mathcal{U}$), then there exists a neighborhood $p \in V \underset{\text{open}}{\subset} \mathbb{P}^n$ and $G_0, G_1, \dots, G_m$ hom. polyn. (same degree) & no common zero on V, s.t. $\varphi_{|V} = [G_0 : \dots : G_m]_{|V}$

Then $\forall$ $x \in \underbrace{\mathcal{U} \cap V}_{\substack{\text{non-empty open} \\ \text{since } \mathbb{P}^n \text{ is irreducible}}}$, we have $[G_0(x) : \dots : G_m(x)] = [F_0(x) : \dots : F_m(x)]$

$$\iff \operatorname{rk} \begin{bmatrix} G_0(x), & \dots, & G_m(x) \\ F_0(x) & \dots & F_m(x) \end{bmatrix} < 2 \iff \forall i,j : \underbrace{F_i(x)G_j(x) = F_j(x)G_i(x)}_{\text{holds } \forall x \in \mathcal{U}\cap V} \quad (*)$$

non-empty open and hence dense in $\mathbb{P}^n$

It follows, that $F_i G_j \overset{\text{as polynomials}}{=} F_j G_i \in K[x_0, \dots, x_n]$

At least one $F_i \neq 0$ (as polynomial). Wlog $F_0 \neq 0$.

$K[x_0, \dots, x_n]$ factorial ring $\Rightarrow$ $F_0 = h_1^{d_1} \cdot \ldots \cdot h_r^{d_r}$, h_i irreducible.

Claim 2: $F_0 \mid G_0$. Indeed, it is enough to show $h_k^{d_k} \mid G_0$ $\forall$ $k = 1, \dots, r$.

Since $F_0, F_1, \dots, F_m$ do not have a common factor, $\exists$ F_i, s.t. $h_k \nmid F_i$.

By $(*)$ $h_k^{d_k} \mid F_0 \cdot G_i = F_i G_0 \Longrightarrow h_k^{d_k} \mid G_0$ ($h_k \nmid F_i$)

It follows, $G_0 = H \cdot F_0$ for some polynomial $H \in K[x_0, \dots, x_n]$.

From $(*)$ it follows that $\forall\, i = 0, 1, \ldots, m$: $\quad G_i = H F_i$.

~~Then $\varphi(p)$~~ ~~Note that $H \neq 0$ on~~

Then $\quad \varphi(p) = [\, \cancel{H(p)}\; G_0(p) : G_1(p) : \ldots : G_m(p)\,] =$

$$= [\, H_0(p) F_0(p) : \ldots : H_m(P) F_m(p)\,] = [\, F_0(p) : F_1(p) : \ldots : F_m(p)\,]$$

Note that $H(p) \neq 0$, otherwise $G_0(p) = \ldots = G_m(p) = 0$.

Hence, Claim 1 is proven and

$$\forall\, p \in \mathbb{P}^4 : \quad \varphi(p) = [\, F_0(p) : F_1(p) : \ldots : F_m(p)\,].$$